

Probabilistic Entailment and Reasoning with Inconsistencies

Abstract. We investigate an approach for drawing logical inference from inconsistent premisses. The main idea in this approach is that the inconsistencies in the premisses should be interpreted as uncertainty of the information. We propose a mechanism, based on Kinght's [13] study of inconsistency, for revising an inconsistent set of premisses to a minimally uncertain, probabilistically consistent one. We will then generalise the probabilistic entailment relation introduced in [15] for propositional languages to first order case to draw logical inference from a probabilistic set of premisses. We will then argue how this combination can allow us to limit the effect of uncertainty introduced by inconsistent premisses to only the reasoning on the part of the premise set that is relevant to the inconsistency.

Keywords: para-consistent reasoning, probabilistic reasoning, inconsistency

1. Introduction

The treatment of inconsistencies is a long standing issue for mathematical logic. Classical logic comes along with strong built-in consistency assumptions and it follows that the full force of the classical entailment relation is too strong for reasoning with inconsistencies. There are, however, many different motivations for the development of logics that can accommodate inconsistencies. Although limiting the scope of logical inference to only consistent domains fits well with the spirit of what one requires from reasoning in mathematical contexts, there are many contexts where it does not. In particular, we have the case when the context of the reasoning is not assumed to represent some factual property of a structure nor objective facts concerning the real state of things but some not-necessarily-certain information or approximations regarding those facts.

Hence there have been several attempts in the literature to develop logical systems and inference processes that allow for reasoning with inconsistent premisses. The main difference between these attempts arise from the way that the inconsistent evidence is interpreted. One motivation stems from adopting the philosophical position of dialetheism as advocated by Graham Priest [22, 24]. This position is characterised by submitting to the thesis that there are true contradictions. That is to accept that there are sentences which are true and false simultaneously. One way to formalise this view is to develop logics that would allow for evaluating a sentence as both true and false, for example by adopting a three valued logic with truth values $\{0, 1, \{0, 1\}\}$ with truth value $\{0, 1\}$ for the sentences that are assumed to be both true and false. The most notable example of such logical systems is arguably the logic LP [21, 23].

Other motivations can arise from more pragmatic reasons which deal with reasoning

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in non-ideal contexts. Here the inconsistencies are interpreted as a property of the information and are taken to be anomalies that point out errors or shortcomings of the reasoners' information (or maybe communication channels). The idea, however, is that despite this shortcoming it is still useful to have formal systems that allow logical inference from such sets of information without submitting to dialethism. The approaches that arise from this latter motivation can be divided into two groups. The first group aim at developing formal systems with mechanisms for dealing with inconsistent information. These include, amongst others, *discussive logic* [12], *adaptive logic* introduced by Batens [3], Da Costa's *logics of formal inconsistency* [7, 8], Dunn-Belnap four-valued logic [4, 9], and *relevant logic* of Anderson and Belnap [2] and their variants. The second group attempts to approach reasoning with inconsistent premisses by reducing the context of reasoning to a consistent one, for instance by defining the logical consequences on the basis of maximal consistent subsets, as in the logical system of Rescher and Manor [25], or by first revising the inconsistent sets to consistent ones, as in AGM belief revision process [1], and make the reasoning on the basis of this consistent set.

Our approach will fall into this last group and is in line with the view that treats inconsistencies as a property of evidence, pointing to some shortcoming or inadequacy of the information, and defines the logical consequence on the basis of some consistent revision of the premisses. In this setting, the existence of inconsistencies points, foremost, to the unreliability of the information, and hence the revision process shifts the context of reasoning from a set of categorically true premisses to uncertain information, expressed probabilistically. As will become clear shortly, however, our approach to revising inconsistent premisses is radically different from that of AGM.

The idea in an AGM-like belief revision process is that upon receiving some information ϕ that is inconsistent with the current knowledge base, one will first retract the part of the premisses that contradicts this new information and then expand the remaining premise set by adding ϕ . The assumption here, however, is that the new information is always more reliable than the old. This assumption is counter-intuitive in many aspects of reasoning; for example when the context of reasoning consists of statements derived from potentially unreliable sources or processes that are subject to errors. Even more pointed are cases that deal with statements accumulated through different sources and processes which do not necessarily agree. This is the case in almost all applications of reasoning outside mathematics. In many such cases as the information set expands by acquiring new information through possibly conflicting sources and processes, it may very well come to include conflicting and inconsistent evidence without any second order information that warrants discarding parts of the premisses in favour of others. At the same time keeping the evidence set whole will void the possibility of using classical entailment (or other variations of it which still get trivialised in the presence of inconsistencies). In this sense having some inconsistency in a (possibly very large) set of evidence will render it completely useless for reasoning. This has motivated a large body of work that deals with "non-prioritised" belief revision [11].

There are many applications of reasoning, however, in which the inconsistencies should

intuitively affect the reasoning only partially. Consider for example sentences ϕ and ψ that share no syntactic component, and the entailment $\{\phi, \psi, \neg\phi\} \models \neg\psi$, many instances of which are counter-intuitive. For instance, assume ϕ and $\neg\phi$ are acquired through different sources, say S_1 and S_2 , where both sources agree on ψ . Here one would expect the inconsistency to effect the one's evaluation of the reliability of the data and thus produce uncertainty in the information, but what is more, at the same time one might wish to do so in a way that introduces as little uncertainty as possible and only where necessary. This is the motivation for what we shall pursue in this paper and the aspect of the literature we hope to contribute to.

Our approach comes in two parts. First is the revision of an inconsistent set of categorical premisses to a probabilistically consistent set of uncertain premisses. In light of the discussion above this should be done in a way that limits the introduction of the uncertainty to only those premisses that are affected by the inconsistency. The second component will then be an entailment relation that allows reasoning on the basis of the new probabilistically consistent premisses. For these we use the work proposed by Knight, [13, 14] in his study of inconsistency, for defining the revision process. A similar approach has also been taken by Thimm in [26, 27] for measuring the inconsistency of a set of probabilistic assertions (see also Bona [5] and Bona et. al. [6]), and by Potyka and Thimm [20] for inconsistency tolerant reasoning. We will then follow the work developed in Knight [15], Picado [18, 19] and Paris, Picado and Rosefield, [17] on probabilistic entailment for propositional languages, and will extend their work to the first order case.

The rest of this paper is organised as follows. In Section 2 we will set up our notation and preliminaries. In Section 3 we will investigate a revision process for reducing inconsistent sets of sentences to probabilistically consistent, uncertain ones. We will investigate revision of categorical inconsistent sets of sentences in Section 3.1, and the revision of inconsistent probabilistic assertions and prioritised sets of sentences in Section 3.2. In Section 4 we generalise a probabilistic entailment relation of [13] to first order languages. Finally, we will argue, in Section 4.3, how generalisation to multiple thresholds, as suggested in [15], will allow us to limit the effect of inconsistency to only part of the reasoning.

2. Preliminaries and Notation

Throughout this paper we will work with a first order language L with finitely many relation symbols, no function symbols and countably many constant symbols a_1, a_2, a_3, \dots . Furthermore we assume that these constants exhaust the universe. This means, in particular, that we have a name for every element in our universe. Thus a model is a structure M for the language L with domain $|M| = \{a_i \mid i = 1, 2, \dots\}$ where every constant symbol is interpreted as itself. Let RL, SL denote the set of relations and the set of sentences of L respectively.

DEFINITION 2.1. A function $w : SL \rightarrow [0, 1]$ is called a probability function if for every $\phi, \psi, \exists x\psi(x) \in SL$,

- P1. If $\models \phi$ then $w(\phi) = 1$.
- P2. $w(\phi \vee \psi) = w(\phi) + w(\psi) - w(\phi \wedge \psi)$.
- P3. $w(\exists x \psi(x)) = \lim_{n \rightarrow \infty} w(\bigvee_{i=1}^n \psi(a_i))$.

We will denote the set of all probability functions on SL by \mathbb{P}_L .

Let L_{prop} be a *propositional language* with propositional variables p_1, p_2, \dots, p_n . By *atoms* of L_{prop} we mean sentences $At = \{\alpha_i \mid i = 1, \dots, J\}$, $J = 2^n$ of the form

$$p_1^{\epsilon_1} \wedge p_2^{\epsilon_2} \wedge \dots \wedge p_n^{\epsilon_n}.$$

where $\epsilon_i \in \{0, 1\}$ and $p^1 = p$ and $p^0 = \neg p$. By disjunctive normal form theorem, for every sentence $\phi \in SL_{prop}$ there is unique set $\Gamma_\phi \subseteq At$ such that $\models \phi \leftrightarrow \bigvee_{\alpha_i \in \Gamma_\phi} \alpha_i$. It can be easily checked that $\Gamma_\phi = \{\alpha_j \mid \alpha_j \models \phi\}$. Thus if $w : SL_{prop} \rightarrow [0, 1]$ is a probability function then $w(\phi) = w(\bigvee_{\alpha_i \in \Gamma_\phi} \alpha_i) = \sum_{\alpha_i \in \Gamma_\phi} w(\alpha_i)$ as the α_i 's are mutually inconsistent. On the other hand, since $\models \bigvee_{i=1}^J \alpha_i$ we have $\sum_{i=1}^J w(\alpha_i) = 1$. So the probability function w will be uniquely determined by its values on the α_i 's, that is by the vector $(w(\alpha_1), \dots, w(\alpha_J)) \in \mathbb{D}^{L_{prop}}$ where $\mathbb{D}^{\mathbb{L} \setminus \mathbb{x}_1} = \{\vec{x} \in \mathbb{R}^J \mid \vec{x} \geq 0, \sum_{i=1}^J x_i = 1\}$. Conversely if $\vec{a} \in \mathbb{D}^{L_{prop}}$ we can define a probability function $w' : SL_{prop} \rightarrow [0, 1]$ such that $(w'(\alpha_1), \dots, w'(\alpha_J)) = \vec{a}$ by setting $w'(\phi) = \sum_{\alpha_i \in \Gamma_\phi} a_i$. This gives a one to one correspondence between the probability functions on L_{prop} and the points in $\mathbb{D}^{L_{prop}}$, see [16] for more details.

The situation for first order languages is a bit more complicated since defining atoms in a way similar to the propositional case will require the use of infinite conjunctions. Instead, what plays the role similar to the atoms for first order languages, are the state descriptions.

DEFINITION 2.2. Let L be a first order language with the set of relation symbols RL and let $L^{(k)}$ be a sub-language of L with only constant symbols a_1, \dots, a_k . The *state descriptions* of $L^{(k)}$ are the sentences $\Theta_1^{(k)}, \dots, \Theta_{n_k}^{(k)}$ of the form

$$\bigwedge_{\substack{i_1, \dots, i_j \leq k \\ R \text{ } j\text{-ary} \\ R \in RL, j \in \mathbb{N}^+}} R(a_{i_1}, \dots, a_{i_j})^{\epsilon_{i_1, \dots, i_j}}.$$

The following theorem, due to Gaifman [10], provides a similar result to the one we had above, for the case of a first order language L . Let $QFSL$ be the set of quantifier free sentences of L :

THEOREM 2.3. *Let $v : QFSL \rightarrow [0, 1]$ satisfy P1 and P2 for $\phi, \psi \in QFSL$. Then v has a unique extension $w : SL \rightarrow [0, 1]$ that satisfies P1, P2 and P3. In particular if $w : SL \rightarrow [0, 1]$ satisfies P1, P2 and P3 then w is uniquely determined by its restriction to $QFSL$.*

The language $L^{(k)}$ can be thought of as a propositional language with propositional variables $R(a_{i_1}, \dots, a_{i_j})$ for $i_1, \dots, i_j \leq k$, $R \in RL$ and R j -ary. With this in mind, for

$\phi \in QFSL$ let k be an upper bound on the i such that a_i appears in ϕ . Then ϕ can be thought of as a propositional formula in $L^{(k)}$. Then the sentences $\Theta_i^{(k)}$ will be the atoms of $L^{(k)}$ and

$$\phi \leftrightarrow \bigvee_{\Theta_i^{(k)} \models \phi} \Theta_i^{(k)} \quad \text{so} \quad w(\phi) = \sum_{\Theta_i^{(k)} \models \phi} w(\Theta_i^{(k)}).$$

Thus to determine the value $w(\phi)$ we only need to determine the values $w(\Theta_i^{(k)})$ and to require

- (i) $w(\Theta_i^{(k)}) \geq 0$ and $\sum_{i=1}^{n_k} w(\Theta_i^{(k)}) = 1$. (ii) $w(\Theta_i^{(k)}) = \sum_{\Theta_j^{(k+1)} \models \Theta_i^{(k)}} w(\Theta_j^{(k+1)})$.

to ensure that P1 and P2 are satisfied.

DEFINITION 2.4. A set $K = \{w(\phi_i) = a_i \mid i = 1, \dots, n\}$ is probabilistically consistent if there is a probability function $w : SL \rightarrow [0, 1]$ that satisfies the constraints given in K .

3. Probabilistically Consistent Revisions

3.1. Probabilistic Revision of Inconsistent Sets of Sentences

Consider a consistent set of sentences $\Gamma = \{\phi_1, \dots, \phi_n\}$ and let θ be such that $\Gamma \cup \{\theta\}$ is inconsistent. In the setting we shall present here, this inconsistency will be characterised as uncertainty and will thus result in moving to some probabilistically consistent revision of $\Gamma \cup \{\theta\}$, Γ' , i.e., a set Γ' consisting of jointly satisfiable probabilistic statements of the form $w(\phi) = a$ for $\phi \in \Gamma \cup \{\theta\}$.

If a set of sentences Γ is classically consistent then the sentences in Γ can be simultaneously assigned probability one. That is, there are probability functions that assign probability one to all sentences in Γ . This will, however, be impossible for an inconsistent Γ , in which case, the highest probability that can be simultaneously assigned to all sentences of Γ will be strictly less than 1. Following Knight [13, 14, 15] we define:

DEFINITION 3.1. Let \mathbb{A} be a set of probability functions on SL . A set of sentences $\Gamma \subseteq SL$ is ζ -consistent in \mathbb{A} , if there is a probability function $w \in \mathbb{A}$ such that $w(\phi) \geq \zeta$ for all $\phi \in \Gamma$. We say that $\Gamma \subseteq SL$ is ζ -consistent, if it is ζ -consistent in \mathbb{P}_L .

Notice that if $\Gamma \not\models \perp$ then Γ is 1-consistent and if $\Gamma \models \perp$ and is ζ -consistent then necessarily $\zeta < 1$. For $\Gamma = \{\phi_1, \dots, \phi_n\}$ let β_i^Γ , $i = 1, \dots, m \leq 2^n$ enumerate the consistent sentences of the form

$$\bigwedge_{i=1}^n \phi_i^{\epsilon_i}$$

where $\epsilon_i \in \{0, 1\}$, $\phi_i^1 = \phi$ and $\phi_i^0 = \neg\phi$. For a probability function w on SL , let $\vec{w}_\Gamma = (w(\beta_1^\Gamma), \dots, w(\beta_m^\Gamma))$. We will drop the superscript and subscript Γ when it is clear from the context. Next we will give a simple Lemma that plays a crucial role in what follows:

LEMMA 3.2. Take $\phi_1, \dots, \phi_n \in SL$, and let β_1, \dots, β_m enumerate the sentences $\bigwedge_{i=1}^n \phi_i^{\epsilon_i}$ as above and let $v(\beta_i)$ be such that $\sum_{i=1}^m v(\beta_i) = 1$. Then there is a probability function w on SL for which $w(\beta_i) = v(\beta_i)$.

PROOF. It is enough to define w on $QFSL$, the quantifier free sentences of L . Choose any probability function u on SL such that $u(\beta_i) \neq 0$ for $i = 1, \dots, m$ and for each n and state description $\Theta^{(n)}$, of $L^{(n)}$, define $w(\Theta^{(n)}) = \sum_{i=1}^m v(\beta_i)u(\Theta^{(n)} | \beta_i)$.

Then clearly $w(\Theta^{(n)}) \geq 0$, also $\sum_{j=1}^{n_k} w(\Theta_j^{(n)}) = \sum_{j=1}^{n_k} \sum_{i=1}^m v(\beta_i)u(\Theta_j^{(n)} | \beta_i) = \sum_{i=1}^m v(\beta_i) \sum_{j=1}^{n_k} u(\Theta_j^{(n)} | \beta_i) = \sum_{i=1}^m v(\beta_i) = 1$ where the equality before last comes from the fact that u is a probability function. Also

$$\begin{aligned} \sum_{\Theta_j^{n+1} \models \Theta^{(n)}} w(\Theta_j^{(n+1)}) &= \sum_{\Theta_j^{n+1} \models \Theta^{(n)}} \sum_{i=1}^m v(\beta_i)u(\Theta_j^{(n+1)} | \beta_i) \\ &= \sum_{i=1}^m v(\beta_i) \sum_{j=1}^{n_k} u(\Theta_j^{(n+1)} | \beta_i) = \sum_{i=1}^m v(\beta_i)u(\Theta^{(n)} | \beta_i) = w(\Theta^{(n)}). \end{aligned}$$

These ensure that w satisfies P1 and P2 and will thus have a unique extension to SL by Gaifman's Theorem. It is clear that $w(\beta_i) = v(\beta_i)$. ■

PROPOSITION 3.3. Let $\Gamma = \{\phi_1, \dots, \phi_n\} \subset SL$. Set $C_\Gamma = \{\zeta | \Gamma \text{ is } \zeta\text{-consistent}\}$ and $\eta = \sup C_\Gamma$. Then Γ is η -consistent.

PROOF. Take a non-decreasing sequence $\zeta_n \in C_\Gamma$ with $\lim_{n \rightarrow \infty} \zeta_n = \eta$. Since each ζ_n is in C_Γ , there is a probability function w_n on SL such that $w_n(\phi) \geq \zeta_n$ for all $\phi \in \Gamma$. Let β_i enumerate sentences $\bigwedge_{i=1}^n \phi_i^{\epsilon_i}$ and $\vec{w}_n = (w_n(\beta_1), \dots, w_n(\beta_m))$ as above. Since $w_n(\beta_1)$ is a bounded sequence, it has a convergent subsequence, say $w_n^1(\beta_1)$ converging to, say b_1 . Let $\vec{w}_n^1 = (w_n^1(\beta_1), \dots, w_n^1(\beta_m))$ be a subsequence of \vec{w}_n specified by the subsequence $w_n^1(\beta_1)$ (so the first coordinates of \vec{w}_n converge to b_1). The same way $w_n^1(\beta_2)$ is a bounded sequence and has a converging subsequence say w_n^2 converging to, say b_2 . Let $\vec{w}_n^2 = (w_n^2(\beta_1), w_n^2(\beta_2), \dots, w_n^2(\beta_m))$ be the subsequence of \vec{w}_n^1 specified by $w_n^2(\beta_2)$ (so the first coordinates converge to b_1 and second coordinates converge to b_2). Continuing the same way, after m steps, we construct a sequence \vec{w}_n^m that converges to (b_1, b_2, \dots, b_m) . Notice that $\sum_{i=1}^m b_i = \sum_{i=1}^m \lim_{n \rightarrow \infty} w_n^m(\beta_i) = 1$. Thus by Lemma 3.2 there is a probability function w on SL such that $w(\beta_i) = b_i = \lim_{n \rightarrow \infty} w_n^m(\beta_i)$ for all $i = 1, \dots, m$. Then for all $\phi \in \Gamma$

$$\begin{aligned} w(\phi) &= \sum_{\beta_k \models \phi} w(\beta_k) = \sum_{\beta_k \models \phi} b_k = \sum_{\beta_k \models \phi} \lim_{n \rightarrow \infty} w_n^m(\beta_k) \\ &= \lim_{n \rightarrow \infty} \sum_{\beta_k \models \phi} w_n^m(\beta_k) = \lim_{n \rightarrow \infty} w_n^m(\phi) \geq \lim_{n \rightarrow \infty} \zeta_n^m = \eta. \end{aligned}$$
■

DEFINITION 3.4. For a set of sentences $\Gamma \subset SL$ the maximal consistency of Γ , denoted by $mc(\Gamma)$ is defined as $mc(\Gamma) = \max\{\eta \mid \Gamma \text{ is } \eta\text{-consistent}\}$.

LEMMA 3.5. Let \mathbb{P}_L be the set of probability function on SL and $\Gamma = \{\phi_1, \dots, \phi_n\} \subset SL$ with $mc(\Gamma) = \eta$. Then

- (i) there is a fixed subset of Γ , say Γ_1 , such that for every probability function $w \in \mathbb{P}_L$ if $w(\phi) \geq \eta$ for all $\phi \in \Gamma$ then $w(\phi) = \eta$ for all $\phi \in \Gamma_1$.
- (ii) there is a partition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_m$, values $\eta_1 < \dots < \eta_m$ and $\mathbb{P}_L = \mathbb{P}_0 \supseteq \mathbb{P}_1 \supseteq \dots \supseteq \mathbb{P}_n$ such that
 - $\mathbb{P}_i = \{w \in \mathbb{P}_{i-1} \mid w(\phi) \geq \eta_i \text{ for all } \phi \in \Gamma \setminus \bigcup_{j=1}^{i-1} \Gamma_j\}$, $i = 1, \dots, n$ and
 - $\eta_i = \max\{\zeta \mid \Gamma \setminus \bigcup_{j=1}^{i-1} \Gamma_j \text{ is } \zeta\text{-consistent in } \mathbb{P}_{i-1}\}$,
 - for all $w \in \mathbb{P}_i$, $w(\psi) = \eta_i$ for all $\psi \in \Gamma_i$.

PROOF. For (i), suppose not, then for every $\psi \in \Gamma$ there is a probability function w_ψ (not necessarily distinct) such that $w_\psi(\phi) \geq \eta$ for all $\phi \in \Gamma$ and $w_\psi(\psi) > \eta$. Let $w = 1/n \sum_{\psi \in \Gamma} w_\psi$ then for every $\phi \in \Gamma$ we have $w(\phi) = 1/n \sum_{\psi \in \Gamma} w_\psi(\phi) > \eta$ since for every $\phi \neq \psi$ $w_\psi(\phi) \geq \eta$ and $w_\psi(\psi) > \eta$. This is a contradiction with $mc(\Gamma) = \eta$.

For (ii), first for a set of probability function \mathbb{A} and a set of sentences Δ define

$$mc_{\mathbb{A}}(\Delta) = \max\{\zeta \mid \Delta \text{ is } \zeta\text{-consistent in } \mathbb{A}\}$$

where the maximum exists. Next notice that for sets of probability functions \mathbb{P}_i above and any finite set of sentences Δ , $mc_{\mathbb{P}_i}(\Delta)$ is well defined. This follows by an argument similar to that of Proposition 3.3 by noticing that if we restrict the construction in the proof of Proposition 3.3 to some \mathbb{P}_i then the probability function w constructed in the that proof that witnesses the threshold $\sup\{\zeta \mid \Delta \text{ is } \zeta\text{-consistent in } \mathbb{P}_i\}$ will also be in \mathbb{P}_i .

Now, let $\eta_1 = mc_{\mathbb{P}_0}(\Gamma) = mc(\Gamma) = \eta$, Γ_1 as in (i), and let $\eta_2 = mc_{\mathbb{P}_1}(\Gamma \setminus \Gamma_1)$. That is the highest threshold that can be simultaneously satisfied by all sentences in $\Gamma \setminus \Gamma_1$ assuming that all sentences in Γ have probability at least η_1 . With the same argument as in (i), one can show that there is a fixed subset $\Gamma_2 \subset \Gamma - \Gamma_1$ such that $w(\theta) = \eta_2$ for $\theta \in \Gamma_2$ and $w(\theta) \geq \eta_2$ for $\theta \in \Gamma - (\Gamma_1 \cup \Gamma_2)$ for every probability function $w \in \mathbb{P}_2$. Following the same process finitely many times one will be left a partition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_m$ and values η_1, \dots, η_m . ■

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_m$ and $\eta_1 < \dots < \eta_m$ be as in Lemma 3.5 and set

$$\vec{mc}(\Gamma) = (\delta_1, \dots, \delta_n), \text{ where } \delta_j = \eta_k \iff \phi_j \in \Gamma_k$$

Intuitively the values given in $\vec{mc}(\Gamma)$ are the highest probabilities that can be assigned to the sentences in Γ coherently. In the sense that there is no probability function that can assign a probability higher than η_1 to all the sentences in Γ_1 simultaneously and same for η_2 and Γ_2 and so on. In other words if we take $\vec{1} = (1, \dots, 1)$ as an n-vector representing

the assignment of probabilities 1 to all sentences ϕ_1, \dots, ϕ_n (which will be impossible if Γ is inconsistent) then for any probability function w if we set $\vec{w} = (w(\phi_1), \dots, w(\phi_n))$, we have $d(\vec{1}, \vec{m}c(\Gamma)) \leq d(\vec{1}, \vec{w})$ where d is the Euclidean distance¹, thus accounting for $\vec{m}c(\Gamma)$ being the closest we can get to the assumption that all sentences in Γ are true.

DEFINITION 3.6. Let $\Gamma = \{\phi_1, \dots, \phi_n\} \subset SL$ be a consistent set of sentences and $\phi_{n+1} \in SL$ be such that $\Gamma \cup \{\phi_{n+1}\} \models \perp$. The revision of Γ by ϕ_{n+1} is defined as

$$\Gamma' = \{w(\phi_1) = a_1, \dots, w(\phi_n) = a_n, w(\phi_{n+1}) = a_{n+1}\}$$

where $(a_1, \dots, a_n, a_{n+1}) = \vec{m}c(\{\phi_1, \dots, \phi_n, \phi_{n+1}\})$.

Definition 3.6 is intended to capture the idea that the revised assignments of probabilities to the sentences $\phi_1, \dots, \phi_n, \phi_{n+1}$ remain as close as possible to 1, i.e., to assign the highest reliability to the information that is probabilistically consistently possible. We will show the uniqueness of $\vec{m}c$ in the next section for a more general setting.

EXAMPLE 3.7. Let L be a first order language with a single binary relation R and equality. And consider the following sentences that express properties of R as a partial order: $\phi_1 : \forall x \neg R(x, x)$ (anti reflexivity), $\phi_2 : \forall x, y (R(x, y) \rightarrow \neg R(y, x))$ (anti symmetry), $\phi_3 : \forall x \exists y R(x, y)$ (each element has an R -successor), $\phi_4 : \forall x, y ((x \neq y) \rightarrow R(x, y) \vee R(y, x))$ (totality), $\phi_5 : \phi_4 \rightarrow \phi_1 \wedge \phi_2$ (if R is total then it is anti reflexive and anti symmetric) and $\phi_6 : \exists x R(x, x)$ and let $\Gamma = \{\phi_1, \dots, \phi_6\}$. Then Γ is inconsistent. It is easy to check that Γ is $1/2$ -consistent and that for any $\zeta > 1/2$, Γ is not ζ -consistent. So η_1 is $1/2$ and $\Gamma_1 = \{\phi_1, \phi_6\}$. Amongst the probability functions that assign a probability of at least $1/2$ to ϕ_1, \dots, ϕ_6 , the highest η_2 such that probability of all ϕ_2, ϕ_3, ϕ_4 and ϕ_5 is at least η_2 is $3/4$ and $\Gamma_2 = \{\phi_4, \phi_5\}$, continuing the same way $\eta_3 = 7/8$ and $\Gamma_3 = \{\phi_2\}$ and $\eta_4 = 1$, $\Gamma_4 = \{\phi_3\}$.

3.2. Revision of Probabilistic Assertions

Using the revision process described above, one will move, in the presence of inconsistencies, from a set of sentences to one consisting of probabilistic assertion on those sentences. To use this as a process for iterated revision one needs to define the revision process also on the sets of probabilistic assertions. The latter will be more general and include the categorical sets by identifying a set $\{\phi_1, \dots, \phi_n\}$ with the set of probabilistic assertions $\{w(\phi_1) = 1, \dots, w(\phi_n) = 1\}$.

Notice that in revising $\Gamma = \{\phi_1, \dots, \phi_n\}$, with a sentence ϕ_{n+1} , the notion of maximal consistency of $\Gamma \cup \{\phi_{n+1}\}$ represents an attempt to jointly assign probabilities to these sentences while remaining as close as possible to their ‘‘prior probabilities’’ (namely, 1). The approach when dealing with probabilistically inconsistent sets of probabilistic assertions is going to be the same. We shall try to jointly revise the probability assignments while remaining as close as possible to the prior probabilities, which might

¹So $d(\vec{x}, \vec{1}) = \sqrt{\sum_{i=1}^n (x_i - 1)^2}$.

not necessarily be 1 any more. To this end we first generalise the notion of maximal consistency given above.

DEFINITION 3.8. Let $\Gamma = \{w(\phi_1) = a_1, \dots, w(\phi_n) = a_n\}$ be a (possibly inconsistent) set of probabilistic assertions. Minimal change consistency of Γ , $m\vec{c}c(\Gamma)$, is defined as the n -vector

$$\vec{q} \in \{\vec{b} \in [0, 1]^n \mid \text{there is a probability function } W \text{ on } SL \text{ with } W(\phi_i) = \vec{b}_i, i = 1, \dots, n\}$$

for which $d(\vec{q}, \vec{a})$ is minimal, where $\vec{a} = (a_1, \dots, a_n)$ and d is the Euclidean distance.

PROPOSITION 3.9. Let $\Gamma = \{w(\phi_1) = a_1, \dots, w(\phi_n) = a_n\}$ be probabilistically inconsistent. Then there is a unique $\vec{b} \in [0, 1]^n$ such that $m\vec{c}c(\Gamma) = \vec{b}$.

PROOF. Let $\Lambda = \{\vec{x} \in [0, 1]^n \mid \text{There is a probability function } w \text{ on } SL \text{ with } w(\phi_i) = x_i\}$. Then Λ is convex. To see this let $\vec{x}, \vec{y} \in \Lambda$ and $\vec{z} = t\vec{x} + (1-t)\vec{y}$ for some $t \in [0, 1]$. Since $\vec{x}, \vec{y} \in \Lambda$, there are probability functions v, w on SL such that $v(\phi_i) = \vec{x}_i$ and $w(\phi_i) = \vec{y}_i$, $i = 1, \dots, n$. Let $u(\psi) = tv(\psi) + (1-t)w(\psi)$ for all $\psi \in SL$. Then u is a probability function on SL and $u(\phi_i) = \vec{z}_i$ for $i = 1, \dots, n$. Thus $\vec{z} \in \Lambda$. Next notice that if d is the Euclidean distance then $f : [0, 1]^n \rightarrow \mathbb{R}$ defined as $f(\vec{x}) = d(\vec{x}, \vec{a})$ is a convex function and so it has a unique minimum on the convex set Λ . ■

It is immediate from the definition that for a set of categorical sentences Γ , i.e., when $a_1 = \dots = a_n = 1$ then $m\vec{c}c(\Gamma)$ coincides with $m\vec{c}(\Gamma)$. Notice also that for probabilistically consistent $\Gamma = \{w(\phi_1) = a_1, \dots, w(\phi_n) = a_n\}$, the $m\vec{c}c(\Gamma) = \vec{a}$. The process of revising a set of probabilistic assertions $\Gamma = \{w(\phi_1) = a_1, \dots, w(\phi_n) = a_n\}$ with the statement $w(\phi_{n+1}) = a_{n+1}$ is the same as revising a non-probabilistic set of sentences but with $m\vec{c}c(\Gamma \cup \{w(\phi_{n+1}) = a_{n+1}\})$ instead of $m\vec{c}(\Gamma \cup \{\phi_{n+1}\})$.

DEFINITION 3.10. Let $\Gamma = \{w(\phi_1) = a_1, \dots, w(\phi_n) = a_n\}$, where $\{\phi_1, \dots, \phi_n\} \subset SL$ and $\phi_{n+1} \in SL$ be such that $\Gamma \cup \{w(\phi_{n+1}) = a_{n+1}\}$ is probabilistically inconsistent², then the revision of Γ by $w(\phi_{n+1}) = a_{n+1}$ is defined as $\Gamma' = \{w(\phi_i) = q_i \mid i = 1, \dots, n+1\}$ where $\vec{q} = m\vec{c}c(\Gamma \cup \{w(\phi_{n+1}) = a_{n+1}\})$.

One thing worth noting here is that, classically there are different ways to eliminate inconsistencies from a set of sentences. One can, for instance, adopt *any* of its maximal consistent subsets, or eliminate inconsistencies in a number of different ways by deleting different sentences. However, as pointed out above, for a set of categorical sentences $\Gamma = \{\phi_1, \dots, \phi_n\}$, $m\vec{c}c(\Gamma) = m\vec{c}(\Gamma)$. Thus Proposition 3.10 ensures that there is a unique way of probabilistically eliminating inconsistency from a set of sentences in the manner that we propose here.

One can immediately notice that in the revision process described above all the sentences are given the same *priority*. This can be readily relaxed. One can modify the distance used in the definition of $m\vec{c}c$ (or $m\vec{c}$) to account for a higher degree of reliability

²that is there is no probability function that can simultaneously assign these values to the sentences in $\phi_1, \dots, \phi_{n+1}$.

or trust in one or some of the probabilistic assertions that are to be revised. Hence we can revise the definition of minimum change consistency as

DEFINITION 3.11. Let $\Gamma = \{w(\phi_i) = a_i \mid i = 1, \dots, n\}$. Then $m\vec{cc}(\Gamma)$, is the n -vector $\vec{q} \in \{(b_1, \dots, b_n) \mid \text{there is a probability function } W \text{ on } SL \text{ with } W(\phi_i) = b_i, i = 1, \dots, n\}$ for which $m(\vec{q}, \vec{b}) := \sqrt{d_i(q_i - a_i)^2}$ is minimal.

We take the assignment of weights d_i to be a context dependent process. There are different approaches one might take to this. When Γ is taken as the set of probabilistic beliefs of an agent, d_i 's can be regarded as what is referred to as the degrees of entrenchment of the beliefs in ϕ_i , expressing how strongly the agent holds their (probabilistic) belief in ϕ_i compared to their belief in ϕ_j , $j \neq i$. One can achieve the same goal by taking a more detailed approach using some notion of ordinal ranking. To see this take the language $L^{(k)}$ ³, and either let Γ consist of only quantifier free sentences, or let k be large enough that $L^{(k)}$ captures a good approximation of the real world for the context. As described in Section 2, $\phi_i^{(k)}$, $i = 1, \dots, n$, can be viewed as sentences in the propositional language with propositional variables $R_i(a_{j_1}, \dots, a_{j_{s_i}})$ and atoms of this language are the sentences of the form

$$\bigwedge_{\substack{j_1, \dots, j_{s_i} \leq k \\ R \text{ } s_i\text{-ary} \\ R_i \in RL, j \in N^+}} R_i(a_{j_1}, \dots, a_{j_{s_i}})^{\epsilon_{j_1, \dots, j_{s_i}}}$$

Then, given an ordinal ranking on these atoms, expressing what the agent takes to be more likely to be the real world, in a way that contradictions are given rank 0, and the more plausible atoms get assigned a higher ordinal, one can take the coefficients d_i above as the highest rank such that there is an atom of that rank consistent with ϕ_i . That is the highest rank of a possible world, of appropriate size, consistent with ϕ_i . On other contextual consideration one might choose to have the coefficients d_i to represent the reliability of the source or the process from which the information is acquired, etc.

4. Probabilistic Entailment

We started with the problem of drawing logical inference from an inconsistent set of premisses. Following the intuition that inconsistencies in the premisses should be interpreted not as a property of the world but rather as a deficiency of the information, we proposed that the presence of inconsistencies should be understood as an inadequacy and hence uncertainty of the information. With this view inconsistencies should be treated by moving from reasoning in a categorical context to the reasoning in uncertain ones, hence moving from a categorical premisses to probabilistic (-ally consistent) ones. Previous section addressed the issue of how to reduce an inconsistent categorical set of premisses to a consistent set of probabilistic assertions. This however leaves open the

³The language with the same relation symbols as L , say R_1, \dots, R_t but with the domain restricted to $\{a_1, \dots, a_k\}$

question of how one should draw logically valid inferences from these sets of probabilistic assertions. We now move to this question.

The classical entailment relation is defined as a process that preserves the truth (given model theoretically). That is the entailment ensure the truth of the conclusion (in a model) given the truth of the premisses (in that model). In the probabilistic setting the truth can be identified by assignment of probability 1. However, for probabilities less than 1 one has to settle for a weaker notion. The precise nature of this weaker notion seems context dependent but instances of such would be for example one of *reliability* or *acceptability*, which in our setting will be represented by a probability threshold. The entailment relation for the sets of probabilistic assertions would then be defined by ensuring the preservation of this weaker notion. Such an entailment relation has been proposed in [13], by Knight and further studied by Knight [14], Paris [?] and Paris, Picado and Rosefield [17]. In this section we will extend this entailment relation to first order languages and will investigate some of its properties. Analogous to [?, 17] we will give an analysis of this entailment relation in terms of the classical consequence relation and will look briefly at its generalisation to multiple probability thresholds which can be used to limit the pathological effect of inconsistencies only to the relevant part of the premisses. As will be clear shortly, the probabilistic entailment we study provides a spectrum of consequence relations, allowing for reasoning at different degree of reliability or acceptability.

4.1. The $\eta \triangleright_{\zeta}$ Entailment

If we identify truth by the probabilistic threshold 1, the classical consequence relation can be read as *if all the premisses are reliable with threshold 1, then so is the conclusion*. The weakening of this relation in our setting is captured by allowing for thresholds less than 1.

DEFINITION 4.1. [13] Let $\Gamma \subset SL$, $\psi \in SL$ and $\eta, \zeta \in [0, 1]$.

$$\Gamma^{\eta} \triangleright_{\zeta} \psi \iff \text{for all probability functions } w \text{ on } L, \text{ if } w(\Gamma) \geq \eta \text{ then } w(\psi) \geq \zeta$$

The idea here is that as long as one is in the position to assign to each of the sentences in Γ a probability of at least η , one is also in the position to assign a probability of at least ζ to the sentence ψ . The intuition for defining such a probabilistic entailment is more evident when $\eta = \zeta$ are interpreted as the thresholds for acceptance. In this situation the entailment relation $\Gamma^{\eta} \triangleright_{\eta} \psi$ can be read as: as long as we are prepared to *accept* all the sentences in Γ we are bound to *accept* ψ . There are situations, however, where the context of reasoning justifies different threshold for the assumptions and conclusion. It can then be immediately observed that for the right value of η this will avoid explosion on inconsistent premisses, for example $\{\phi, \neg\phi, \psi\}^{1/2} \not\triangleright_{1/2} \neg\psi$. To be more precise, one can avoid the trivialisation of the entailment relation $\eta \triangleright_{\zeta}$ as long as one chooses $\eta \leq mc(\Gamma)$. Thus for the rest of this section we shall restrict ourselves to $\eta \in [0, mc(\Gamma)]$ whenever we make a reference to $\Gamma^{\eta} \triangleright_{\zeta}$.

4.2. Some Properties of $\eta \triangleright_\zeta$

Following [17] we now show some elementary properties of $\eta \triangleright_\zeta$ for the first order case.

PROPOSITION 4.2. *For any $\Gamma = \{\phi_1, \dots, \phi_n\} \subset SL$ and $\psi \in SL$,*

- (i) $\Gamma^\eta \triangleright_0 \psi$.
- (ii) For $\zeta > 0$, $\Gamma^1 \triangleright_\zeta \psi \iff \Gamma \models \psi$.
- (iii) For $\eta > mc(\Gamma)$, $\Gamma^\eta \triangleright_1 \psi$.
- (iv) For $\zeta > 0$, $\Gamma^0 \triangleright_\zeta \psi \iff \models \psi$.

PROOF. Parts (i) and (iii) are immediate from the definition. Notice that classical valuations on L are themselves probability functions. Thus for consistent Γ , $\Gamma^1 \triangleright_\zeta \psi$ implies that $v(\psi) \geq \zeta$ for all valuations v for which $v(\Gamma) = 1$. Since $\zeta > 0$ this implies that $v(\psi) = 1$ and thus $\Gamma \models \psi$. If Γ is inconsistent then (ii) follows trivially. Conversely suppose $\Gamma \models \psi$ and $w(\Gamma) = 1$. Let β_i , $1 \leq i \leq m$, enumerate sentences of the form $\bigwedge_{i=1}^n \phi_i^{\epsilon_i}$ where $\epsilon_i \in \{0, 1\}$ and $\phi_i^1 = \phi_i$ and $\phi_i^0 = \neg \phi_i$. Then for any β_i such that $w(\beta_i) > 0$ we have $\beta_i \models \phi_i$ for all $1 \leq i \leq n$ since otherwise we will have $w(\phi_i) = \sum_{\beta_j \models \phi_i} w(\beta_j) < 1$. So $\beta_i \models \bigwedge \Gamma$ and since $\bigwedge \Gamma \models \psi$, $w(\psi) \geq w(\bigwedge \Gamma) = w(\bigwedge \Gamma) = \sum_{\beta_j \models \bigwedge \Gamma} w(\beta_j) = 1 \geq \zeta$ as required. For (iv), if $\not\models \psi$ then there is a valuation v for which $v(\psi) = 0$. Since v is also a probability function and $v(\phi) \geq 0$ for all $\phi \in \Gamma$, $\Gamma^0 \triangleright_\zeta$ will fail for any $\zeta > 0$. Conversely if $\Gamma^0 \triangleright_\zeta \psi$ fails then there is a probability function w for which $w(\psi) < \zeta \leq 1$ and thus $\not\models \psi$. ■

PROPOSITION 4.3. *Assume that $\Gamma^\eta \triangleright_\zeta \psi$. Then*

- (i) If $\tau \geq \eta$ and $\nu \leq \zeta$, then $\Gamma^\tau \triangleright_\nu \psi$.
- (ii) if $\tau \geq 0$ and $\eta + \tau, \zeta + \tau \leq 1$, then $\Gamma^{\eta+\tau} \triangleright_{\zeta+\tau} \psi$

PROOF. (i) is immediate from the definition. For (ii) suppose that $\Gamma^{\eta+\tau} \triangleright_{\zeta+\tau} \psi$ failed. Thus there is a probability function w for which $w(\phi) \geq \eta + \tau$ for all $\phi \in \Gamma$ but $w(\psi) < \zeta + \tau$. If $w(\psi) < \zeta$ we will have that $\Gamma^\eta \triangleright_\zeta \psi$ fails. Otherwise let $\gamma \geq 0$ be such that

$$\gamma < \zeta < \gamma + (\zeta + \tau - w(\psi)).$$

Let β_i enumerate all the sentences of the form $\bigwedge_{i=1}^n \phi_i^{\epsilon_i} \wedge \psi^{\epsilon_{n+1}}$. Pick a β_i such that $w(\beta_i) > 0$ and $\beta_i \not\models \psi$ (such a β_i exists otherwise we should have $w(\psi) = 1$ and $\Gamma^{\eta+\tau} \triangleright_{\zeta+\tau} \psi$ will hold). Define

$$v(\beta_k) = \begin{cases} w(\beta_k) \cdot (\gamma/w(\psi)) & \text{if } \beta_k \models \psi, \\ w(\beta_k) & \text{if } \beta_k \not\models \psi, \beta_k \neq \beta_i, \\ w(\beta_i) + w(\psi) - \gamma & \text{if } \beta_k = \beta_i \end{cases}$$

so $\sum_{k=1}^{2^{n+1}} v(\beta_k) = 1$. Using Lemma (3.2), we can find a probability function w' on SL such that $w'(\beta_i) = v(\beta_i)$ for $i = 1, \dots, 2^n$. Then we have $w'(\psi) = \sum_{\beta_i \models \psi} w'(\beta_i) = \sum_{\beta_i \models \psi} w(\beta_i) \cdot \gamma/w(\psi) = \gamma$ and for $\phi \in \Gamma$ we have $w(\phi) - w'(\phi) \leq \sum_{\beta_i \models \phi \wedge \psi} w(\beta_i)(1 - \gamma/w(\psi)) \leq w(\psi) - \gamma$ since all other β_k increase in probability under w' , $w'(\phi) \geq \eta + \tau - (w(\psi) - \gamma) > \eta$. So we have $w'(\phi_i) > \eta$ while $w'(\psi) = \gamma < \zeta$ which contradicts $\Gamma^\eta \triangleright_\zeta \psi$. ■

The next result shows that the entailment relation $\eta \triangleright_\zeta$ does not depend on the choice of language. More precisely, let L_1, L_2 be finite first order languages and such that $\Gamma \subset SL_1 \cap SL_2$ and $\psi \in SL_1 \cap SL_2$, then $w_1(\psi) \geq \zeta$ for every probability function w_1 on SL_1 such that $w_1(\Gamma) \geq \eta$ if and only if $w_2(\psi) \geq \zeta$ for every probability function w_2 on SL_2 such that $w_2(\Gamma) \geq \eta$.

PROPOSITION 4.4. *The relation $\eta \triangleright_\zeta$ is language invariant.*

PROOF. Let $\Gamma \subset SL$ and $\psi \in SL$ such that $\Gamma^\eta \triangleright_\zeta \psi$ in the context of the language L , i.e., for every probability function w on SL if $w(\Gamma) \geq \eta$ then $w(\psi) \geq \zeta$. It is enough to show that if L' is a language such that $L \subset L'$ then for every probability function w' on SL' , if $w'(\Gamma) \geq \eta$ then $w'(\psi) \geq \zeta$ and conversely.

For the forward direction assume that w' is a probability function on SL' such that $w'(\Gamma) \geq \eta$ but $w'(\psi) < \zeta$. Let w be the restriction of w' to SL . Then w will be a probability function that agrees with w' on Γ and ψ and thus $\Gamma^\eta \triangleright_\zeta \psi$ will fail in the context of the language L . Conversely let w be a probability function on SL such that $w(\Gamma) \geq \eta$ but $w(\psi) < \zeta$. Let $\Gamma = \{\phi_1, \dots, \phi_n\}$ and as before let β_i enumerate the sentences of the form $\bigwedge_{i=1}^n \phi_i^{\epsilon_i} \wedge \psi^{\epsilon_{i+1}}$ and we have that $w(\psi) = \sum_{\beta_i \models \psi} w(\beta_i) < \zeta$. Since $L \subset L'$, we have $\beta_i \in SL'$ and since w is a probability function we have that $\sum_{i=1}^{2^{n+1}} w(\beta_i) = 1$. Using lemma 3.2, we can find a probability function w' on SL' with $w'(\beta_i) = w(\beta_i)$. With the notation of lemma 3.2, for $\phi \in \Gamma$,

$$w'(\phi) = \sum_{i=1}^{2^{n+1}} w(\beta_i) u(\phi | \beta_i) = \sum_{\beta_i \models \phi} w(\beta_i) = w(\phi) \geq \eta$$

and

$$w'(\psi) = \sum_{i=1}^{2^{n+1}} w(\beta_i) u(\psi | \beta_i) = \sum_{\beta_i \models \psi} w(\beta_i) = w(\psi) < \zeta.$$

Hence $\Gamma^\eta \triangleright_\zeta \psi$ fails in the context of language L' . ■

With this in place we can now talk about making logical inference from an inconsistent set of premisses. Let $\Gamma \models \perp$ and $\eta = mc(\Gamma)$. As pointed out in the previous section η can be regarded as the highest threshold of reliability that can be jointly satisfied by all sentences in Γ . One can then devise an spectrum of entailment relations $\eta \triangleright_\zeta$ for $\zeta \in [0, 1]$. Given the intuition we started with it seems more reasonable however to limit the spectrum to $\zeta \in [\eta, 1]$. With $\zeta = 1$ one would be effectively define the logical inference from Γ as the set of sentences that will be categorically true if one was to accept sentences in Γ with the highest possible threshold. Similarly values of $\zeta \in [\eta, 1)$ can correspond to more relaxed criteria of acceptability for what can be considered as a consequence of Γ . Given a set of sentences $\Gamma \subset SL$, let $\eta = mc(\Gamma)$ and define $\Gamma \approx_\zeta \psi \iff \Gamma^\eta \triangleright_\zeta \psi$. Notice that if we denote the set of consequences of Γ at reliability degree ζ by C_Γ^ζ then for $\zeta \leq \delta$ we have $C_\Gamma^\delta \subseteq C_\Gamma^\zeta$.

4.3. Generalising to Multiple Thresholds; $\vec{\eta} \triangleright_{\zeta}$

What is missing from this picture so far is the promise of an entailment relation that can limit the effect on inconsistencies to only the part of reasoning that is relevant to them. For this we should look at the $\vec{m}c(\Gamma)$ and generalise the entailment relation to allow for multiple thresholds.

DEFINITION 4.5 ([15]). Let $\Gamma = \{\phi_1, \dots, \phi_n\} \subset SL$, $\psi \in SL$ and $\vec{\eta} \in [0, 1]^n, \zeta \in [0, 1]$. Define

$$\Gamma^{\vec{\eta}} \triangleright_{\zeta} \psi \iff \text{for all probability functions } w \text{ on } L, \\ \text{if } w(\phi_i) \geq (\vec{\eta})_i \text{ for } i = 1, \dots, n \text{ then } w(\psi) \geq \zeta.$$

Let $\Gamma = \{\phi_1, \dots, \phi_n\}$ be an inconsistent set of sentences. Notion of $\vec{m}c(\Gamma)$, introduced in the previous section, was meant to capture the highest probability that can be simultaneously assigned to sentences in Γ , capturing the highest degree of reliability that one can consider for them. In this sense the entailment $\Gamma^{\vec{m}c(\Gamma)} \triangleright_{\zeta}$ allows us to relax the notion of logical consequence of a set Γ by considering not only the models in which sentences in Γ hold categorically (of which there are none since Γ is inconsistent) but extend to probabilistic models in which sentences in Γ are as reliable as possible. The next result shows how this can be employed to limit the effect of inconsistencies to only reasoning from the relevant part of the premisses.

PROPOSITION 4.6. *Let $\Pi \subseteq \mathcal{P}(\Gamma)$ be the set of maximally consistent subsets of Γ and let $\Delta = \bigcap \Pi$, then $(\vec{m}c(\Gamma))_i = 1$ whenever $\phi_i \in \Delta$. Indeed Δ can be regarded as the part of Γ to which the inconsistency is irrelevant.*

PROOF. Let $\Delta = \{\phi_1, \dots, \phi_t\} = \bigcap \Pi$, $\Gamma' = \Gamma \setminus \Delta = \{\psi_1, \dots, \psi_n\}$ and $\vec{m}c(\Gamma') = \vec{\eta} = (\eta_1, \dots, \eta_n)$. Then there is a probability function u on SL such that $u(\psi_i) = \eta_i$. Let $\alpha_i, i = 1, \dots, m \leq 2^n$, enumerate all the satisfiable sentences of the form $\bigwedge_{k=1}^n \psi_k^{\epsilon_k}$ and sentences $\beta_{i, \vec{\epsilon}}$ enumerate all the sentences of the form

$$\beta_{i, \vec{\epsilon}} = \alpha_i \wedge \bigwedge_{k=1}^t \phi_k^{(\vec{\epsilon})^k}.$$

Define $v(\beta_{i, \vec{\epsilon}}) = u(\alpha_i)$ if $\vec{\epsilon} = \vec{1}$ and $v(\beta_{i, \vec{\epsilon}}) = 0$ otherwise. Then $\sum_{i, \vec{\epsilon}} v(\beta_{i, \vec{\epsilon}}) = \sum_{i=1}^{2^m} v(\alpha_i) = 1$.

By Lemma 3.2 then there is a probability function w on SL such that $w(\beta_{i, \vec{\epsilon}}) = v(\beta_{i, \vec{\epsilon}})$. Fix the order of sentences in Γ as $\{\psi_1, \dots, \psi_n, \phi_1, \dots, \phi_t\}$, and let $\vec{w} = (w(\psi_1), \dots, w(\psi_n), w(\phi_1), \dots, w(\phi_t))$. We now show that $\vec{w} = \vec{m}c(\Gamma)$. This will complete the proof since $w(\psi_i) = u(\psi_i) = \eta_i, i = 1, \dots, n$ and $w(\phi_j) = 1, j = 1, \dots, t$. So w assigns probability 1 to all sentences in Δ .

To see that $\vec{w} = \vec{m}c(\Gamma)$ let v be any probability function on SL that disagrees with w on some sentences in Γ so $\vec{v} = (v(\psi_1), \dots, v(\psi_n), v(\phi_1), \dots, v(\phi_t)) \neq \vec{w}$, then

$$d(\vec{v}, \vec{1}) = \sqrt{\sum_{i=1}^n (\vec{v}_i - 1)^2 + \sum_{j=n+1}^{n+t} (\vec{v}_j - 1)^2} > \sqrt{\sum_{i=1}^n (\eta_i - 1)^2 + 0}$$

$$= \sqrt{\sum_{i=1}^n (\vec{w}_i - 1)^2 + \sum_{j=n+1}^{n+t} (\vec{w}_j - 1)^2} = d(\vec{w}, \vec{1})$$

where the strict inequality comes from uniqueness of $\vec{m}c(\Gamma')$ and the fact that $\vec{v} \neq \vec{w}$. ■

This ensures that $\vec{m}c(\Gamma)$ assigns probability 1 to all sentences that are not affected by the inconsistency in Γ . That is for any $\psi \in SL$ such that $\Delta \models \neg\psi$, $\Gamma \vec{m}c \triangleright_{\zeta} \psi$ for all $\zeta > 0$ and if $\Delta \models \psi$ then $\Gamma \vec{m}c \triangleright_{\zeta} \psi$ for all $\zeta \in [0, 1]$.

EXAMPLE 4.7. Consider L_1 and L_2 to be disjoint languages with $L = L_1 \cup L_2$ and let $\Gamma_1 \subset SL_1$ and $\Gamma_2 \subset SL_2$ and $\Gamma = \Gamma_1 \cup \Gamma_2 \subset SL$. Let $\Gamma_1 = \{\phi_1, \dots, \phi_n\}$ be inconsistent with $\vec{m}c(\Gamma_1) = (\eta_1, \dots, \eta_n)$ and assume that $\Gamma_2 = \{\psi_1, \dots, \psi_m\}$ is consistent and so $\vec{m}c(\Gamma_2) = (1, \dots, 1)$. Then taking $\Gamma = \{\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_m\}$ in this fixed order, we have $\vec{m}c(\Gamma) = (\eta_1, \dots, \eta_n, 1, \dots, 1)$. Define $\Gamma \vec{m}c \approx_{\zeta} \psi \iff \Gamma \vec{m}c \triangleright_{\zeta} \psi$. Again, we have a spectrum of entailment relations from the set Γ each at a different degree of reliability in $[0, 1]$. Now for $\theta \in SL_2 \subset SL$ we have $\Gamma \vec{m}c \approx_{\zeta} \theta \iff \Gamma_2 \models \theta$, thus reducing the inference on sentences of L_2 where the relevant knowledge is consistent to the classical inference, hence limiting the pathological effect of the inconsistency only to inferences on sentences of L_1 where the knowledge is inconsistent.

5. Conclusion

One approach to deal with inconsistencies is motivated by reasoning in non-ideal contexts and is based on the assumption that the inconsistent evidence does not point out the inconsistencies of the reality under investigation but point to an inconsistent evaluation of facts. Receiving contradictory information should thus cast doubts on those evaluations. In this view, receiving some piece of information ϕ while having $\neg\phi$ in our knowledge base has the effect of changing the (categorical or probabilistic) evaluation of ϕ (and thus $\neg\phi$). In case of categorical knowledge (with truth values of zero or one), this means moving from categorical belief in ϕ or $\neg\phi$ to some uncertain evaluation of them and in case of probabilistic knowledge this would entail re-evaluation of the probabilities. This approach, as we followed here, is based on two assumptions,

- the inconsistencies are identified with the uncertainty that they induce in the information set
- the information is assumed to be as reliable as possibly allowed by the consistency considerations.

Hence, receiving inconsistent information will change the context of reasoning from a categorical one to an uncertain one that is expressed probabilistically. We built upon the work introduced by Knight [13, 14, 15] and argued that it is possible to do so in a way that allows limiting the pathological effect of an inconsistency to the part of the reasoning that is relevant to it.

How the change induced by the inconsistency is carried out in the information set depends on one's approach to the weighting of the new information with respect to the old information. For example, if we take the new information to be infinitely more reliable than the old, we will end up with the same retraction and expansion process as in the AGM. But as we have seen, one can also devise the change in a manner that allows a wider range of epistemic attitudes towards the new information in comparison to the old. Since the inconsistencies will reduce our categorical knowledge to probabilistic one, any inference based on such knowledge will essentially be probabilistic. We then studied a probabilistic entailment relation on propositional languages, introduced by Knight, and showed that it can be extended to the first order case in a very straight forward manner. The idea on this entailment relation is to generalise the classical consequence relation from a relation that preserves the truth to one that preserves, or more precisely ensures, some degree of reliability.

It is also worth mentioning that one can choose a different route altogether and deal with the inconsistent evidence by adopting a richer language in which the source, and/or reliability of information is also coded in the information. Thus, for example, ϕ received from source S is replaced by $(\phi)_S$ to the effect that "according to S , ϕ ". In this approach receiving ϕ_S and $(\neg\phi)_{S'}$ poses no contradiction any more, while contradictory information from the same source has the effect of reducing the reliability of the source. The evaluation of information should then depend on the reliability of the sources. This approach however can, at least to some extent, be covered by our setting. The simplest case we discussed corresponds to dealing with equally reliable pieces of information. And the notion of maximal consistency can be regarded as the highest reliability that one can assign to a source that gives contradictory information. The case of prioritised evidence can cover dealing with (possibly inconsistent) information from sources that have different reliabilities. The approach given here, however, has the advantage of avoiding unnecessary complication of the language.

Of course our notion of "closeness" when revising the inconsistent theories into probabilistically consistent ones can be subject to debate. The use of Euclidean distance was motivated by trying to choose the closest values for all sentences simultaneously. It would be interesting to investigate if other notions of "closeness" can improve this approach.

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